Compact Equivalent Circuit Model for the Skin Effect

S. Kim and D. P. Neikirk

Department of Electrical and Computer Engineering
Electrical Engineering Research Laboratory
The University of Texas at Austin
Austin, Texas 78712

Abstract

Rules for determining a compact circuit model consisting of four resistors and three inductors to accurately predict the skin effect have been developed. The circuit is easily constructed from the geometry, producing a response that matches exact results over a frequency range from dc to very high frequencies.

Introduction

The skin effect has been extensively studied, although many methods treat the skin effect at high frequencies only, failing to predict the properties of transmission lines at low frequencies. Time domain analysis for digital signal propagation requires models covering the entire frequency range from dc to $1/\tau_{rise}$, where τ_{rise} is the rise time of the signal. Skin effect lumped circuit models in which the elements are frequency independent have been used [1-3], but tend to produce very large ladder circuits. Yen *et al.* [3] introduced a more compact circuit model, but this method failed to accurately capture the skin effect at high frequencies and did not establish clear rules governing the choice of component values. Here we present a modification of Yen's method using simple rules for selecting the values of resistors and inductors for a four deep ladder circuit model. The equivalent circuit accurately models the skin effect in circular cross section conductors up to a frequency corresponding to a 100 skin depth radius conductor. We also show equivalent circuits for the series impedance per unit length for coax and twin lead, including both proximity and skin effects.

Compact Frequency Independent Circuit Model for Round Wires

For wide bandwidth digital signals on normal lossy transmission lines, since conductor loss increases as frequency increases, the transmission line acts somewhat like a low pass filter. As the signal propagates along the line and high frequency components are attenuated, the effective bandwidth decreases, and hence an "electrically short" length becomes longer. This has been used to reduce the size of lumped ladder models for long transmission lines through the use of non-uniform lumping [4, 5]. This approach can also be used to generate compact circuit models for the skin effect. Yen *et al.* [3] introduced a constant resistance ratio (RR) to try and capture this low pass characteristic. Figure 1 shows a schematic illustration of our compact ladder model. Here a circular cross section conductor is divided into four concentric rings, each ring represented by one ladder section. For the resistance values each ring is chosen so that $R_i/R_{i+1} = RR$, $i = 1, 2, 3$, where RR is a constant to be determined.

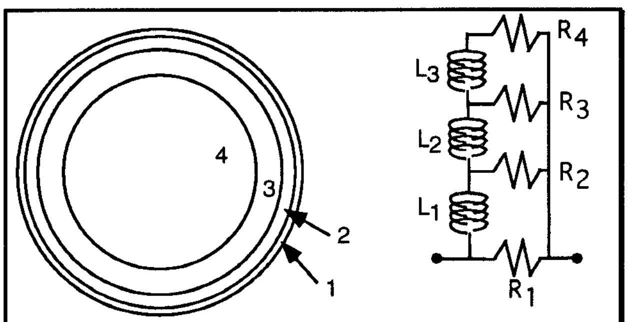

Figure 1: Schematic illustration of four ladder compact circuit model; drawing is to scale for $\alpha_R = 10$, giving RR = 1.66.

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We first require that the dc resistance of the ladder be equal to the actual dc resistance of the conductor R_{dc}, and also take the first resistance to be

$$R_1 = \alpha_R \cdot R_{dc} \ . \tag{1}$$

For a four section ladder these two constraints lead to the requirement

$$(RR)^3 + (RR)^2 + RR + (1 - \alpha_R) = 0 \ . \tag{2}$$

Thus, for a given selection of α_R, the resistance ratio RR is fixed by solution of this cubic equation. The inductance values are determined in a similar fashion, with $L_i / L_{i+1} = LL$, $i = 1, 2$, again requiring that the low frequency inductance of the ladder be equal to the actual low frequency internal inductance (L_{lf}) of the wire. Using $L_1 = L_{lf}/\alpha_L$ this leads to the constraint that

$$\left(\frac{1}{LL}\right)^2 + \left(1 + \frac{1}{RR}\right)^2 \frac{1}{LL} + \left(\left[\frac{1}{RR}\right]^2 + \frac{1}{RR} + 1\right)^2 - \alpha_L\left(\left[1 + \frac{1}{RR}\right]\left[\left\{\frac{1}{RR}\right\}^2 + 1\right]\right)^2 = 0 \tag{3}$$

This equation can be solved for LL, once RR has been obtained from the solution of the cubic polynomial (eq. 2) and α_L has been selected. To match the high frequency resistance of the conductor, the resistance of the outermost ring (R_1) is most critical; we have found that if the maximum frequency of interest is ω_{max}, α_R should be chosen so that

$$\alpha_R = 0.53 \frac{\text{wire radius}}{\delta_{\mathbf{max}}} \ , \tag{4}$$

where

$$\delta_{\max} = \sqrt{\frac{2}{\omega_{\mathbf{max}} \mu_o \sigma}} \ . \tag{5}$$

To ensure the frequency response from dc to ω_{max} is well modeled, we have found that the inductance values must be chosen using

$$\alpha_L = 0.315 \alpha_R \ . \tag{6}$$

Once the dimensions of the conductor and ω_{max} are specified, all the values of the components in the ladder are fixed, and the response of this circuit from dc to ω_{max} will match the skin effect.

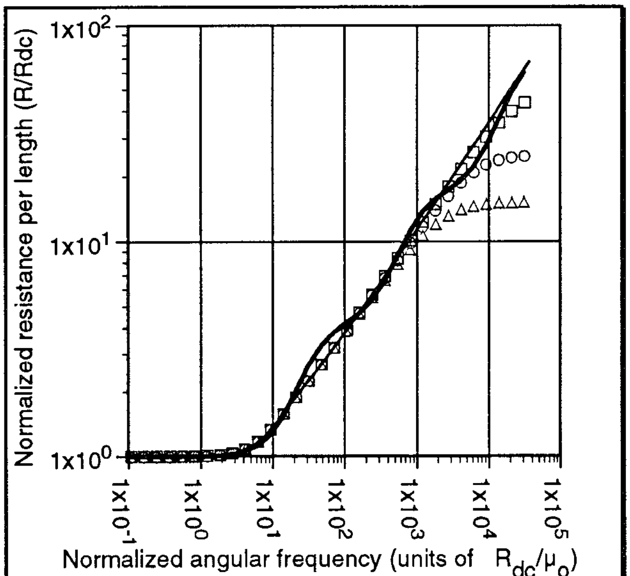

Figure 2: Comparison between uniform lumping and compact model. Thin line: exact result from solving Helmholtz equation; thick line: four ladder circuit modeling; □: uniform ladder, 100 sections; ○: uniform ladder, 50 sections; Δ: uniform ladder, 30 sections.

Figure 2 illustrates the advantage of using this method over a uniform lumping method. For instance, over 100 uniform R-L sections are necessary to represent the response of a 50 skin-depth radius circular conductor to similar levels of accuracy as a four section ladder using the procedure discussed above. We have found that for $5 \leq \alpha_R \leq 50$ the local error is not worse than 15%, and with α_R up to 100, the error is still not worse than 25%. This corresponds to wires with radii between about 10 and 100 skin depths at ω_{max} for less than 15% error, and up to 200 skin depths at 25% error.

Coax Example

This approach can be used to model a coaxial line including skin effect in both the center and shield conductors. The exact solution for the series impedance per unit length, including skin effect can be found in [6]. The series impedance equivalent circuit is shown in Fig. 3a, using the rules presented above applied to both inside and outside conductor, one ladder for the inner conductor using $L_{lf} = \mu_o/8\pi$, and another for the outer shield using [7]

$$L_{lf}^{outer} = \frac{\mu_o}{2\pi}\left[\frac{c^4 \ln(c/b)}{\left(c^2 - b^2\right)^2} + \frac{b^2 - 3c^2}{4\left(c^2 - b^2\right)}\right] . \quad (7)$$

Figure 3b shows the result of circuit modeling for a maximum frequency up to that corresponding to a 100 skin-depth inner conductor radius. Both resistance and inductance are in excellent agreement with the exact results.

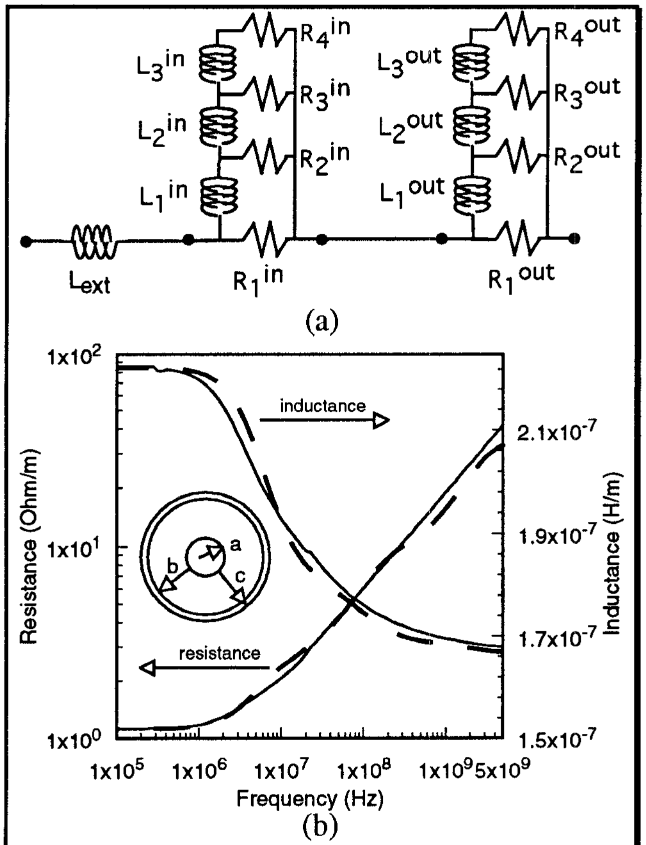

Figure 3: Coaxial line impedance calculation; (a) series impedance skin effect equivalent circuit. (b) Calculated total series impedance for a coaxial line with inner conductor radius $a = 0.1$ mm, shield radius $b = 0.23$ mm, shield thickness 0.02 mm ($c = 0.25$ mm). Thin line: exact result; dashed line: compact circuit model, $f_{max} = 5$ GHz.

Twin lead Example

For twin lead, when two conductors are very closely separated both the skin effect and proximity effect cause series resistance to increase. To approximate the proximity effect with a simple equivalent circuit, we find the fraction of the circular conductor ζ that contributes half the flux at high frequency (Fig. 4a):

$$\zeta = \frac{\theta}{4\pi} = \frac{1}{\pi}\sin^{-1}\left(\sqrt{1-\left(\frac{r}{d}\right)^2}\right) . \quad (8)$$

Two ladders with values determined using the rules presented above are constructed, with one ladder weighted by $1/\zeta$ (representing the inner face of the conductors), the other weighted by $1/(1 - \zeta)$ (representing the outer faces of the conductors); they are then connected in parallel, as shown in Fig. 4b. Figure 4c shows the equivalent circuit compared to a conformal mapping method [8]. The conformal mapping method over-estimates the low frequency inductance; at low frequencies our circuit model is actually in better agreement with exact results.

Conclusion

Rules for determining a compact circuit model consisting of four resistors and three inductors that accurately predicts the skin effect have been developed. Each element value can be easily calculated from the geometry and conductivity to cover a frequency range from dc to very high frequencies. Since this model contains only frequency independent elements, it is easy to implement in conventional circuit simulators. This model can also be used to capture proximity effects, and can be generalized for application to rectangular conductor geometries.

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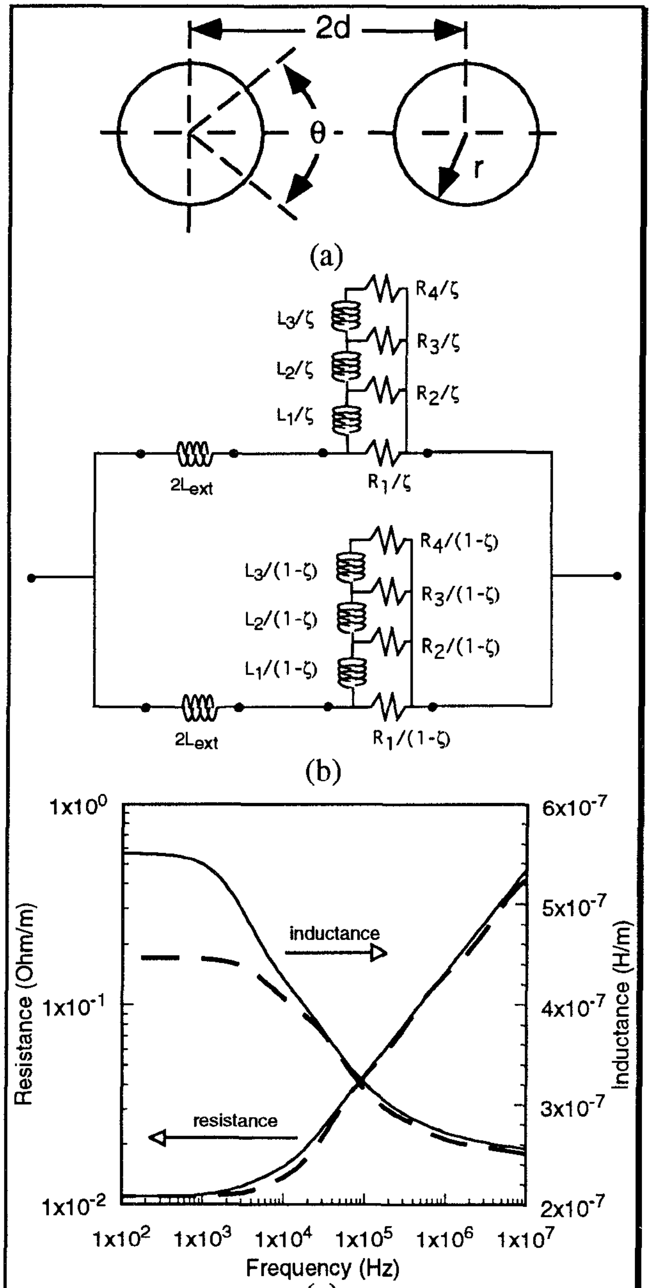

Figure 4: Example of twin lead equivalent circuit including external inductance and the proximity effect. (a) geometry of twin lead; θ and ζ represent half the flux coupling the two conductors; (b) equivalent circuit; (c) sample calculation for r = 1 mm, d = 1.2 mm: solid line: conformal mapping [8]; dashed line: circuit model.

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